ON THE JACOBIAN CONJECTURE

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Abstract

We show that the Jacobian conjecture of the two dimensional case is true.

1. Introduction

Let $T = (P(x, y), Q(x, y)) : \mathbb{C}^2(x, y) \to \mathbb{C}^2(\xi, \eta)$ be a polynomial mapping. If $T \in Aut(\mathbb{C}^2)$, its Jacobian $J(T)$ is a non-vanishing polynomial. Then $J(T)$ is a non-zero constant.

The Jacobian conjecture says that if $J(T)$ is a non-zero constant, then $T \in Aut(\mathbb{C}^2)$.

In 1993, Kaliman showed that to prove the Jacobian conjecture, we may assume that every level curve of $P$ is irreducible [8]. In other words, there is a polynomial automorphism $S$ such that all level curves of $P(S)$ are irreducible when $T = (P, Q)$ is a polynomial mapping with $J(T) = c$, where $c$ is a non-zero constant.
In 2011, we claimed that the Jacobian conjecture is affirmative by using the Kaliman’s result essentially but incompletely [2].

In this paper, we will prove completely by the same method in a little refined manner.

2. Preliminaries and Definitions

Definition 2.1. We say that \( P(x, y) \) is nonsingular if \( \{P_x = P_y = 0\} = \emptyset \) and that a nonsingular polynomial \( P(x, y) \) satisfies the condition (K) when its every level curve is irreducible.

Remark 2.2. \( P(x, y) = x(xy - 1) \) is a nonsingular polynomial but \( \{P = 0\} \) is reducible [8].

Definition 2.3. Let \( P(x, y) \) satisfy the condition (K). We consider the following linear partial differential equation where \( u(x, y) \) is an unknown function

\[
J(P, u) = g(x, y). \tag{1}
\]

We say that the problem (A) is solvable when for every entire function \( g(x, y) \), equation (1) has a global solution.

Remark 2.4. We remark that the problem (A) is not solvable when some level curve is not simply connected (Lemma 5.1). So if the problem (A) is solvable, every level curve of \( P \) is simply connected and the Jacobian conjecture follows rather easily (Theorem 5.3). Conversely, if every level curve of \( P \) is simply connected, then the problem (A) is solvable (Proposition 4.7).

Remark 2.5. In 1994, Bartolo et al. [3] showed that there is a polynomial which satisfies the condition (K) whose level curve is not simply connected. Such a concrete example was shown in [5]. So the problem (A) is not solvable always.
Definition 2.6. Let \( P(x, y) \) satisfy the condition (K). When there is a polynomial \( Q(x, y) \) such as \( J(P, Q) = c \), where \( c \) is a non-zero constant, we say that \( P \) satisfies the condition (J) and \( Q \) is an associated polynomial with \( P \).

Remark 2.7. When the problem (A) is solvable, \( P \) satisfies the condition (J) from Corollary of Theorem 4 in [7], because \( P \) satisfies the condition (K) and every level curve of \( P \) is simply connected (Remark 2.4).

We will solve the problem (A) under the condition (J).

3. Basic Facts

Let \( P(x, y) \) be a nonsingular polynomial. Following proposition is a summarized statement from Suzuki [10, p. 242].

Proposition 3.1. Let \( P(x, y) \) be a nonsingular polynomial. Then except for at most finite number of level curves, every level curve \( \{ P = a \} \) is irreducible and the same topological type \((g, n)\), that is, the normalization of \( \{ P = a \} \) has a genus \( g \) and \( n \) punctured boundaries.

Remark 3.2. If \( P \) satisfies the condition (K) and \( g = 0 \) in the same notation of above proposition, every level curve is simply connected from Theorem A in [9] and then the problem (A) is solvable. So almost all level curves of polynomial in Remark 2.5 have genus \( g > 0 \) and no one is simply connected from Corollary of Theorem 4 in [10].

It is easy to see the following proposition from Proposition 3.1.

Proposition 3.3. Let \( P(x, y) \) satisfy the condition (K). Let \( \{ P = a_i \} \) be an exceptional level curve in the sense of Proposition 3.1 and exceptional values be \( \{ a_i \}_{i=1,\ldots,k} \). When \( a \neq a_1, \ldots, a_k \), there is a sufficiently small disk \( U(a) \) with the center \( a \) without \( a_1, \ldots, a_k \) such that \( E_a := \{(x, y) \in P(x, y) \} \) is considered topologically as \( U(a) \times R \), where \( R \) is a \((g, n)\)-type Riemann surface.
Proposition 3.4. Let $P$ satisfy the condition $(J)$ and $Q$ is an associated polynomial with $P$. For every $a \in \mathbb{C}$ if we take a sufficiently small disk $U(a)$, we can consider $E_a := \{(x, y); U(a) \ni P(x, y)\}$ as a fiber space such as $E_a = (E_a, P, U(a))$ and there is a global holomorphic section $\sigma_a$ of $E_a$ such as $\{Q(x, y) = r\} \cap E_a$, where $\{Q(x, y) = r\} = \sigma$ is irreducible in $\mathbb{C}^2$ and $P$ takes every value $b \in \mathbb{C}$ on $\sigma$.

Proof. Since $Q$ is a nonsingular polynomial, almost all level curves are irreducible by Proposition 3.1. Since $T = (P, Q)$ is a locally biholomorphic mapping and the image of $T$ is a Zariski open set, the complement of the image of $T$ contains at most finite points of $\mathbb{C}^2(\xi, \eta)$. So, it is easy to see that there is a complex number $r$ such as above. 

Remark 3.5. Note that $\sigma_a$ in Proposition 3.4 is simply connected. When $a \neq b$, $\sigma_a \cap \sigma_b = \emptyset$ or $\sigma_a \cap \sigma_b$ is a simply connected open set on $\sigma$.

4. Solve the Problem (A)

We will solve the problem (A) under the condition $(J)$. To construct a global solution of (1), we prepare lemmas about local solution.

Since (1) is a nonsingular linear partial differential equation, there is a unique (single valued) holomorphic solution $u_a$ near $\sigma_a$ having any given holomorphic initial data on $\sigma_a$ by Cauchy-Kowalevskaya’s theorem and the assumption that $\sigma_a$ is simply connected.

When we restrict equation (1) to $\{P = b\}$, where $b \in U(a)$, it is a holomorphic 1-form such as $du = \frac{g}{P_y} dx = -\frac{g}{P_x} dy$, where $P_y dx + P_y dy = 0$. In other words, $\{P = b\}$ is a characteristic curve of (1). So above local solution $u_a$ can analytically prolong along any path on $\{P = b\}$.
Lemma 4.1. Let $E_a$ be same in Proposition 3.3. Then above solution $u_a$ can prolong free in $E_a$, that is, $u_a$ can (analytically) prolong along any path from a point $p$ of $\sigma_a$ to a point $q$ in $E_a$.

Proof. We remark that the universal covering space $\tilde{E}_a$ can be considered as $U(a) \times \tilde{R}$. Since $u_a$ can be considered as a single valued function on $\tilde{E}_a$ and holomorphic near $\sigma_a$, $u_a$ is holomorphic on $\tilde{E}_a$ by Hartogs theorem. It means that $u_a$ can prolong free in $E_a$. □

Lemma 4.2. We use the same notations in above lemma. Let $u$ be a function element which is obtained by an analytic prolongation of $u_a$ along a path $l$ from a point $p \in \sigma_a$ to a point $q$ where $l \subset E_a$ and $q \in \{P = b\}$. Set $\sigma_a \cap \{P = b\} = p'$. Then there is a path $l'$ from $p'$ to $q$ where $l' \subset \{P = b\}$ such that we have a function element $u$ when we prolong $u_a$ along $l'$.

Proof. From the proof of Lemma 4.1, $u_a$ can extend a holomorphic function on $\tilde{E}_a$. Let $\tilde{l}$ be the lift of $l$ and project it to $\{P = b\}$ as $l'$. It is easy to see that we have a function element $u$ when we prolong $u_a$ along $l'$. □

Lemma 4.3. Let $E_a, E_b, a \neq b$ be same as Proposition 3.3 and $\sigma_a \cap \sigma_b \neq \emptyset$. If $u_a = u_b$ on $\sigma_a \cap \sigma_b$, then $u_a \cup u_b$ can prolong free in $E_a \cup E_b$.

Proof. We have a conclusion easily from Lemma 4.2. □

From Lemmas 4.1 and 4.3, following corollary is obtained easily.

Corollary 4.4. Let $a_i$ be an exceptional value and $U(a_i)$ be a disk with the center $a_i$ without other exceptional values. Let $U^*(a_i) = \{U(a_i) - a_i\}$ and $E_{a_i}^* = \{U^*(a_i) \ni P(x, y)\}$. Then $u_{a_i}$ can prolong free in $E^*(a_i)$. 

Lemma 4.5. Let $a_i$ and $U(a_i)$ be the same in above corollary. Then $u_{a_i}$ can prolong free in $E_{a_i}$, where $E_{a_i} = \{U(a_i) \ni P(x, y)\}$.

Proof. By Tuber Neighborhood Theorem (see [6, p. 67], for example), there is an open Stein neighborhood $U$ of $S = \{P = a_i\}$ in $E_{a_i}$ and a homotopy of holomorphic map $f_s : U \to U$, $s \in [0, 1]$ such that $f_0$ is the identity map on $U$, $f_s|S$ is the identity map on $S$ for all $s \in [0, 1]$, and $f_1(U) = S$.

At first, let $l$ be an arbitrary path from $p$ to $q \in S$ such as $l \subset U$ and $u$ be a function element at $p$ which satisfies equation (1). When we consider $f_s(l)$, they are homotopy curves such that $f_0(l) = l$, $f_1(l) \subset S$ and these terminal points are $q$. So $u$ can prolong along the path $l$ since $S$ is a characteristic curve and its function element at $q$ satisfies equation (1) from the principle of permanence of functional relations.

At second, let $l$ be a path from $p \in \sigma_{a_i}$ to $q \in S$ in $E_{a_i}$. From above discussion and Corollary 4.4, $u_{a_i}$ can prolong to $q$ along $l$.

Finally, let $l$ be an arbitrary path in $E_{a_i}$ from $p \in \sigma_{a_i}$ to $q$. Then from above discussion and Corollary 4.4, it is easy to see that $u_{a_i}$ can prolong along $l$ to $q$. □

Theorem 4.6. When $P$ satisfies the condition $(J)$, the problem $(A)$ is always solvable.

Proof. Let $\sigma_j$ be a countable covering of $\sigma$ and its projection by $P$ is $U_j$ such as $U(a)$ in Lemma 4.1 or $U(a_j)$ in Corollary 4.4. And we use notations $E_j$ similarly as Lemmas 4.1 and 4.5. Let $u_j^0$ be a holomorphic initial data on $\sigma_j$ and $u_j$ be all function elements obtained by the prolongation along every path in $E_j$ having the initial data $u_j^0$. 
When $\sigma_j \cap \sigma_k = \sigma_{jk} \neq \emptyset$, $s \in [0, 1]$ on $\sigma_j \cap \sigma_k$ is a holomorphic function near $\sigma_{jk}$ and $J(u_k^0 - u_j^0) = 0$. Then it is easy to see that for every $b \in U_j \cap U_k$, $u_k^0 - u_j^0$ is a constant on $\{P = b\}$ and we can define $\varphi_{jk} \in O(\sigma_j \cap \sigma_k)$ such as $u_k^0 - u_j^0 = \varphi_{jk}(P)$ near $\sigma_{jk}$.

From the definition, it is easy to see that $\varphi_{jk} = -\varphi_{kj}$. And when $\sigma_j \cap \sigma_k \cap \sigma_l = \sigma_{jkl} \neq \emptyset$, it is easy to see that $\varphi_{jk} + \varphi_{kl} + \varphi_{lj} = 0$ on $\sigma_{jkl}$.

Then $\{\varphi_{jk} : \sigma_j \cap \sigma_k\}$ is a Cousin-I data. Since $\sigma$ is a Stein manifold, one can find functions $\{\varphi_j \in O(\sigma_j)\}$ such as $\varphi_j - \varphi_k = \varphi_{jk}$ when $\sigma_j \cap \sigma_k \neq \emptyset$.

We set $u_j = u_j^0 + \varphi_j(P)$. It prolongs free in $E_j$. Let $E_j \cap E_k \neq \emptyset$ and $\sigma_j \cap \sigma_k = \sigma_{jk} \neq \emptyset$. It is easy to see that $u_j^0 + \varphi_j(P) - (u_k^0 + \varphi_k(P)) = 0$ on $\sigma_{jk}$. And then $u_i \cup u_j$ can prolong free in $E_j \cup E_k$.

By the successive discussion, $\bigcup_{j=1}^\infty u_j$ prolongs free in $C^2 = \bigcup_{j=1}^\infty E_j$ and it is a global solution by the monodromy theorem.

**Proposition 4.7.** When $P$ satisfies the condition $(K)$ and every level curve is simply connected, the problem $(A)$ is solvable.

**Proof.** Let us recall Theorem 2.1 in [1]. To prove above proposition, we must check the condition (a) and (c) in the statements of Theorem 2.1. From the condition $(K)$, it is easy to see that the conditions (a) and (c) are satisfied.

**5. Conclusion**

**Lemma 5.1.** If the problem $(A)$ is solvable, then $\{P = a\}$ is simply connected for every $a \in C$. 
Proof. If some level curve $S = \{ P = a \}$ is not simply connected, then there is a holomorphic 1-form $a(x) \, dx = b(y) \, dy$, where $b(y) = a(x) \left( -\frac{P_y}{P_x} \right)$ whose integral on $S$ is a multi-valued function by Behnke-Stein theorem [4]. If we set $a(x)P_y = -b(y)P_x$, then it represents a holomorphic function $g_0$ on $S$. Since $\mathbb{C}^2$ is a Stein domain, there is an entire function $g(x, y)$ such that $g|_S = g_0$. From the explanation in outset of Section 4, $du = \frac{g_0}{P_y} \, dx = a(x) \, dx$ on $S$. It is easy to see that equation (1) for that $g$ has no global solution. This contradicts the assumption. \qed

Next proposition is a special case of Theorem 3.3 in [1].

Proposition 5.2. Let $P(x, y)$ satisfy the condition (K) and every level curve is simply connected and $u(x, y)$ be an entire function. Then $(P, u) \in \text{Aut}(\mathbb{C}^2)$ if and only if $u$ satisfies the equation $J(P, u) = \varphi(P)$ where $\varphi$ is an entire function such as $\varphi \neq 0$.

Theorem 5.3. If the polynomial map $T = (P(x, y), Q(x, y))$ satisfies $J(P, Q) \neq 0$, then $T \in \text{Aut}(\mathbb{C}^2)$.

Proof. By virtue of Kaliman, we may assume that $P$ satisfies the condition (K). From Theorem 4.6, Lemma 5.1 and Proposition 5.2, we conclude $T \in \text{Aut}(\mathbb{C}^2)$. \qed

Following proposition is an explicit version of Jung [11].

Proposition 5.4. If $(P, Q) \in \text{Aut}_{\text{alg}}(\mathbb{C}^2)$, $P = x + \psi (cy + \varphi(x))$, $Q = cy + \varphi(x)$, where $\varphi, \psi$ are polynomials of one complex variable and $c$ is a non-zero constant. And the inverse is true.
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Proof. Let \((P, Q), (P', Q') \in Aut_{\text{alg}}(\mathbb{C}^2)\) such as \(J(P, Q) = J(P, Q') = c\). Then \(J(P, Q' - Q) = 0\). As the same reason of the proof of Theorem 4.6, \(Q' - Q = \varphi(P)\), where \(\varphi\) is a polynomial of one complex variable.

Let \((P', Q'), (P, Q') \in Aut_{\text{alg}}(\mathbb{C}^2)\) such as \(J(P', Q') = J(P, Q') = c\). Then similarly as above, \(P' - P = \psi(Q')\), where \(\psi\) is a polynomial of one complex variable. Let \(J(P', Q') \in Aut_{\text{alg}}(\mathbb{C}^2)\) such that \(J(P', Q') = c\). Then \(P' = P + \psi(Q'), \ Q' = Q + \varphi(P)\). Now we set \(P = x, Q = cy\). Then we have above the conclusion.

Let \(P = x + \psi(cy + \varphi(x)), \ Q = cy + \varphi(x)\), where \(c\) is a non-zero constant. Then \(J(P, Q) = c\). Also, \((P, Q) \in Aut_{\text{alg}}(\mathbb{C}^2)\). \(\square\)

References


